UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS

GCE Advanced Level

MARK SCHEME for the October/November 2010 question paper for the guidance of teachers

9231 FURTHER MATHEMATICS

9231/01 Paper 1, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes must be read in conjunction with the question papers and the report on the examination.

- CIE will not enter into discussions or correspondence in connection with these mark schemes.

CIE is publishing the mark schemes for the October/November 2010 question papers for most IGCSE, GCE Advanced Level and Advanced Subsidiary Level syllabuses and some Ordinary Level syllabuses.

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

<u>Penalties</u>

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

1 $1+\left(\frac{dy}{dx}\right)^2 = 1+(\frac{1}{2}(e^{2x}-e^{-2x}))^2 = \frac{1}{4}(e^{2x}+e^{-2x})^2$ — M1A1 — expression simplified

Length $= \int_0^{\frac{1}{2}} \frac{1}{2}\left(e^{2x}+e^{-2x}\right)dx = \frac{1}{4}\left[\left(e^{2x}-e^{-2x}\right)\right]_0^{\frac{1}{2}}$ — M1 — integrate

$= \frac{1}{4}\left(e^1-e^{-1}\right)-\frac{1}{4}\left(e^0-e^0\right)=\frac{e^2-1}{4e}$ **AG** — A1 — cao — [4]

2 nth term is $\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right)$ — M1A1

$$S_N = \frac{1}{2}\left[\begin{array}{l}\left(\frac{1}{N}-\frac{1}{N+2}\right)+\left(\frac{1}{N-1}-\frac{1}{N+1}\right)+\left(\frac{1}{N-2}-\frac{1}{N}\right)+\ldots \\ \left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{1}-\frac{1}{3}\right)\end{array}\right]$$ — M1 — sum of terms

$= \frac{1}{2}\left[\frac{3}{2}-\frac{1}{N+2}-\frac{1}{N+1}\right]$ — A1 — after cancellation — [4]

Limit = ¾ — B1√ — [1]

3 Area $= \int_1^4\left(x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)dx = \left[\frac{2}{3}x^{\frac{3}{2}}-2x^{\frac{1}{2}}\right]_1^4 = 8/3$ — B1

$$\bar{y} = \frac{\frac{1}{2}\int_1^4 (x-2+\frac{1}{x})dx}{A} = \frac{\frac{1}{2}\left[\frac{x^2}{2}-2x+\ln x\right]_1^4}{A}$$

M1 — use of $\frac{\frac{1}{2}\int y^2 dx}{A}$

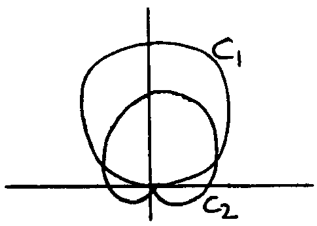
M1 — integrate

A1 — correct

Final answer:

$\frac{3}{8}\left(\ln 2+\frac{3}{4}\right)$ or $\frac{3}{16}\left(\ln 4+\frac{3}{2}\right)$ or $\frac{3}{8}\ln 2+\frac{9}{32}$ etc (ACF) — A1 — [5]

4	$n = 0$: $7^1 + 5^3 = 132$ which is divisible by 44	B1		
	Assume $7^{2k+1} + 5^{k+3}$ is divisible by 44	B1		
	Consider $7^{2(k+1)+1} + 5^{(k+1)+3} = 7^2 7^{2k+1} + 5.5^{k+3}$	M1	$(k + 1)$ th term	
	$= 49(7^{2k+1} + 5^{k+3}) - 44.5^{k+3}$	M1	in appropriate form	
	which is divisible by 44	A1	convincing argument	[
	Alternative solution for final three marks:			
	Consider $(7^{2k+3} + 5^{k+4}) - (7^{2k+1} + 5^{k+3})$	M1		
	$= 48(7^{2k+1} + 5^{k+3}) - 44.5^{k+3}$	M1	in appropriate form	
	which is divisible by 44	A1	convincing argument	
5	$I_{n+2} = [-(1-x)^{n+2}\cos x] - \int (n+2)(1-x)^{n+1}\cos x\,dx$	M1A1		
	$= (1 + (n+2)) + (n+2)[((1-x)^{n+1}\sin x) + \int (1-x)^n \sin x\,dx]$	M1	integrate by parts again	
	$I_{n+2} = 1 - (n+1)(n+2)\,I_n$ **AG**	A1		[4]
	$I_6 = 1 - 5 \times 6I_4$; $I_4 = 1 - 4 \times 3I_2$; $I_2 = 1 - 1 \times 2I_0$	M1		
	$I_0 = \int_0^1 \sin x\,dx = 1 - \cos 1$	B1		
	$I_6 = 1 - 30(1 - 12(1 - 2I_0)) = 0.0177$	M1A1		[4]
	OR			
	$I_0 = 1 - \cos 1$	B1		
	$I_2 = 2\cos 1 - 1$	M1	(use of RF)	
	$I_4 = 13 - 24\cos 1$	A1		
	$I_6 = 0.0177$	A1	cao	
	Accept decimal versions			
6	$\begin{pmatrix} 1 & 2 & -1 & \alpha \\ 0 & -1 & 1 & -2\alpha \\ 0 & -3 & 4 & -2-2\alpha \\ 0 & 1 & -3 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 & \alpha \\ 0 & -1 & 1 & -2\alpha \\ 0 & 0 & 1 & 4\alpha - 2 \\ 0 & 0 & 0 & 6\alpha - 6 \end{pmatrix}$	M1A1		
	Dim = 4 $\Rightarrow \alpha \neq 1$ **AG**	A1		[3]
	$a + 2b - c = 0$ $2a + 3b - c = 0$ Show $a = b = c = 0$ $2a + b + 2c = 0$ $b - 3c = 0$	M1	attempt to solve	
	Linearly independent and dim R(T) not 4: basis	A1		[2]
	$a + 2b - c = p$ $2a + 3b - c = 1$ Attempt to find a, b, c in terms of q or p $2a + b + 2c = 1$			
	$b - 3c = q$	M1A1		
	$6p + q = 3$	A1		[3]
	<u>Alternative solution:</u>			
	Use row operations as in (i)	M1		
	Final column $\begin{pmatrix} p \\ 1-2p \\ 4p-2 \\ 6p+q-3 \end{pmatrix}$	A1		
	$6p + q = 3$	A1		

Qn	Answer	Mark	Notes	Total
7	$y=\dfrac{1}{x+1}\quad \therefore x=\dfrac{1-y}{y}$	M1	use in given cubic eq	
	Gives $6y^3-7y^2+3y-1=0$ **AG**	A1		
	$n=1$: given expression = sum of roots = 7/6	B1		
	$n=2$: $\sum\dfrac{1}{(\alpha+1)^2}=\left(\sum\dfrac{1}{(\alpha+1)}\right)^2-2\sum"\alpha\beta"=\dfrac{13}{36}$	B1		[2]
	From cubic in y,			
	$6\sum\left(\dfrac{1}{\alpha+1}\right)^3-7.\dfrac{13}{36}+3\left(\dfrac{7}{6}\right)-3=0$	M1		
	$\sum\left(\dfrac{1}{\alpha+1}\right)^3=73/216$	A1		[2]
	LHS = $\sum\left(\dfrac{(\beta+1)(\gamma+1)(\alpha+1)}{(\alpha+1)^3}\right)$	M1		
	$=\left(\dfrac{1}{6}\right)^{-1}\times\dfrac{73}{216}$	M1	recognise product of roots	
	= 73/36 **AG**	A1		[3]
8	**(i)** $1+\sin\theta=3\sin\theta\Rightarrow\sin\theta=\dfrac{1}{2}$	M1		
	$\left(\dfrac{3}{2},\dfrac{\pi}{6}\right)$ and $\left(\dfrac{3}{2},\dfrac{5\pi}{6}\right)$	A1	(both)	[2]
	(ii) 	B1 B1 B1	circle cardioid behaviour at origin cardioid closed and symmetry	[3]
	(iii) Subtract integrands	M1		
	$2\times\frac{1}{2}\int_{\pi/6}^{\pi/2}(3-4\cos 2\theta-2\sin\theta)\,d\theta$	M1		
	$=\left[3\theta-2\sin 2\theta+2\cos\theta\right]_{\pi/6}^{\pi/2}$	M1A1		
	$=\pi$ **AG**	A1		[5]
	<u>Alternative:</u> Area inside C_1:			
	$2\times\dfrac{1}{2}\int_{\pi/6}^{\pi/2}9\sin^2\theta\,d\theta=\dfrac{9}{2}\left[\theta-\dfrac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/2}$	M1		
	$=\dfrac{9}{2}\left(\dfrac{\pi}{3}+\dfrac{\sqrt{3}}{4}\right)$	A1		

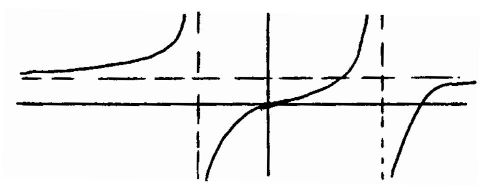
	Area inside C_2:			
	$2\times\frac{1}{2}\int_{\pi/6}^{\pi/2} 1+2\sin\theta+\frac{1}{2}(1-\cos 2\theta)\,\mathrm{d}\theta$			
	$=\left[\frac{3\theta}{2}-2\cos\theta-\frac{1}{4}\sin 2\theta\right]_{\pi/6}^{\pi/2}$	M1		
	$=\left(\frac{\pi}{2}+\frac{9\sqrt{3}}{8}\right)$	(A1	if not earned earlier)	
	Subtraction	M1		
	Required area = π **AG**	A1		[5]
9	$(3-\lambda)\left[(2-\lambda)(3-\lambda)-1\right]+1(-(3-\lambda))=0$	M1	characteristic equation	
	$(3-\lambda)(\lambda-1)(\lambda-4)=0$	M1	factorise	
	$\lambda = 1, 3, 4$	A1		
	$\begin{pmatrix}3-\lambda & -1 & 0\\ -1 & 2-\lambda & -1\\ 0 & -1 & 3-\lambda\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$			
	Solve for $\lambda = 1$: (1, 2, 1)	M1A1		
	Solve for $\lambda = 3$: (1, 0, –1)	A1		
	Solve for $\lambda = 4$: (1, –1, 1)	A1		[7]
	$\mathbf{M}=\begin{pmatrix}1 & 1 & 1\\ 2 & 0 & -1\\ 1 & -1 & 1\end{pmatrix}$	B1√	eigenvectors as columns (except $\begin{pmatrix}0\\0\\0\end{pmatrix}$)	
	$\mathbf{D}=\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 8\end{pmatrix}$	M1A1√	ft on eigenvalues	[3]
10	$\cos 5\theta = c^5-10c^3s^2+5cs^4$	M1A1	use of de Moivre for $(c+\mathrm{i}s)^5$	
	$\sin 5\theta = 5c^4s-10c^2s^3+s^5$	A1		
	$\tan 5\theta = \frac{t^5-10t^3+5t}{1-10t^2+5t^4}$ **AG**	M1A1	intermediate step needed	[5]
	$\tan 5\theta = 0 \Rightarrow \theta = \frac{n\pi}{5}$	M1		
	Solutions $\tan\frac{n\pi}{5}$ for n = 1, 2, 3, 4	A1	justify values of n	[2]
	Roots $\pm\tan\frac{\pi}{5}, \pm\tan\frac{2\pi}{5}$	B1		
	Product of these roots = 5	M1		
	$\tan\frac{\pi}{5}\tan\frac{2\pi}{5}=\sqrt{5}$	A1		[3]

11	$z' = y + xy'$	B1		
	$z'' = 2y' + xy''$	B1		
	Obtain result	B1		
	Auxiliary equation: $m^2 + 4 = 0$: $m = \pm 2i$	M1		
	CF: $A\cos 2x + B\sin 2x$	A1		
	PI: $z = ax^2 + bx + c$			
	Differentiate twice and substitute	M1		
	$a = 2, b = 0, c = 3$	A1		
	GS: $z = A\cos 2x + B\sin 2x + 2x^2 + 3$	A1√	their CF + their PI	
	$y = 0, x = \frac{1}{2}\pi$: $(z = 0)$ gives $A = \frac{\pi^2}{2} + 3$	B1		
	$z' = -2A\sin 2x + 2B\cos 2x + 4x$	M1		
	$y' = -2, x = \frac{\pi}{2}$: $(z' = -\pi)$ gives $B = \frac{3\pi}{2}$	A1		
	$y = \frac{1}{x}\left(\left(\frac{\pi^2}{2} + 3\right)\cos 2x + \frac{3\pi}{2}\sin 2x + 2x^2 + 3\right)$	A1		[9]

12 EITHER

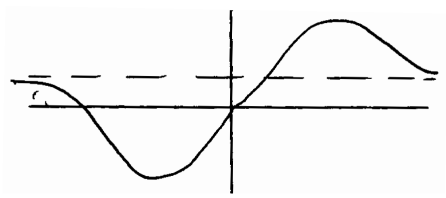
(i)	$y' = 0 \Rightarrow (x^2 - 2x + \lambda)(2x + 2\lambda) - (x^2 + 2\lambda x)(2x - 2) = 0$	M1		
	$\Rightarrow \ldots \Rightarrow (\lambda + 1)x^2 - \lambda x - \lambda^2 = 0$	A1		
	Hence at most 2 values of x and at most 2 stationary points	A1		[3]
(ii)	For 2 real distinct roots, $\lambda^2 > 4(\lambda + 1)(-\lambda^2)$	M1	use of discriminant	
	$\lambda^2(5 + 4\lambda) > 0 \therefore \lambda > -\frac{5}{4}$ **AG**	A1		[2]
(iii)	Vert. asymptotes when $x^2 - 2x + \lambda = 0$	M1		
	$b^2 - 4ac > 0 \Rightarrow 4 - 4\lambda > 0$			
	For two vert. asymp. $\lambda < 1$	A1		[2]
(iv) (a)	$y = 0 \Rightarrow x^2 + 2\lambda x = 0$	M1		
	$\Rightarrow x = 0$ or -2λ	A1	(both)	
(b)	$y = 1$: $x = \frac{\lambda}{2\lambda + 2}$	B1		[3]

(v) (a) $\lambda < -2$: no stat points: 2 vert. asymp



B1	3 branches	
B1	completely correct shape	

(b) $\lambda < 2$: 2 stats points: no vert. asymp



B1	max, min, horiz asymp	
B1	correct shape	[4]

OR

Normal to plane: (2, 3, 4) × (–1, 0, 1) = (3, –6, 3)	M1A1		
r.(1, –2, 1) = d and point (2, 1, 4)	M1	substitute point into plane e	
$d = 4 \quad x - 2y + z = 4$	A1		[

Alternative:

$\left.\begin{array}{l} x = 2 + 2\lambda - \mu \\ y = 1 + 2\lambda \\ z = 4 + 4\lambda + \mu \end{array}\right\\} \quad x + z = 6 + 6\lambda$	M1A1		
$\therefore x + z = 6 + 2(y - 1)$	M1		
$\therefore x - 2y + z = 4$	A1		

$x - 4y + 5z = 12$			
$x - 2y + z = 4$ Solve by eliminating one variable	M1		
Use parameter and express all 3 variables in terms of it	M1		
e.g. $x = 3t - 4$, $y = 2t - 4$, $z = t$			
r = (–4, –4, 0) + t (3, 2, 1)	A1	or equivalent	[3]

Alternative:

Direction of line = $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} = t\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$	M1A1		
Find any point on line e.g. $\begin{pmatrix} -4 \\ -4 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ etc.			
$\therefore \mathbf{r} = \begin{pmatrix} -4 \\ -4 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$	B1		

Line l: **r** = (a, $2a + 1$, –3) + α($3c$, –3, c)			
Plane: $x - 2y + z = 4$			
Distance A to plane:			
$\left\lvert \dfrac{a - 2(2a + 1) - 3 - 4}{\sqrt{6}} \right\rvert = \dfrac{15}{\sqrt{6}}$	M1		
$3a + 9 = 15$	M1	correct use of modulus sign	
$a = 2$	A1		
$\sin\theta = \dfrac{3c + 6 + c}{\sqrt{6}\sqrt{9c^2 + 9 + c^2}}$	M1A1		
$\therefore \dfrac{4c + 6}{\sqrt{6}\sqrt{9 + 10c^2}} = \dfrac{2}{\sqrt{6}}$	M1	solve for c	
$6c^2 - 12c = 0$: $c = 2$	A1		
(Penalise only once for negative values.)			[7]